MA 681, Spring 2016

Assignment 7.

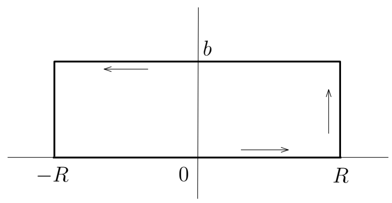
Cauchy Theorem. Cauchy Integral Formula.

This assignment is due Wednesday, March 9. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2bx)dx, \qquad b \in \mathbb{R}, b > 0.$$

(*Hint:* Integrate e^{-z^2} along the path shown in the figure. You can take for granted that $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$. Integrals along the vertical sides should go to zero.)



(2) Evaluate the integral using Cauchy integral formula:

$$\int_{|z-a|=a} \frac{z}{z^4-1}dz,$$

where $a \in \mathbb{R}, a > 1$.

(3) Evaluate the integral using Cauchy integral formula:

$$\int_L \frac{e^z}{z^2+a^2}dz,$$

where $a \in \mathbb{R}$ and $I(L)$ contains the closed disc $|z| \leq a$.

(4) Prove that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi.$$

(*Hint:* Use average value theorem for $f(z) = \cos z$.)

— see next page —

— do not panic —

(5) *Cauchy formula for an unbounded domain.* Let L be a closed rectifiable simple curve, traversed counterclockwise. Let $f(z)$ be a differentiable function on a domain G, where $L \cup E(L)$ is contained in G. (That is, f is differentiable on a neighborhood of L and *outside* of L, but not necessarily inside.) Suppose that

$$\lim_{z\to\infty} f(z) = A.$$

Prove that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = A, \quad \text{if } z \in I(L),$$

and

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = -f(z) + A, \quad \text{if } z \in E(L).$$

(*Hint:* Pass to integration over a large circle $|z| = R$ using Cauchy theorem for multiple contours. Then use Problem 3 of HW6.)

(6) Let $f(z) = \frac{2016z^{12} - z^3 + 3102z^2 + 100}{(z-1)^3(z-2)^4(z-3)^2(z-4)(z-5)^2}$. Evaluate

$$\int_L \frac{f(\zeta)}{\zeta - z_0}d\zeta,$$

if $I(L)$ contains the disc $|z| \leq 5$, and $z_0 \neq 1, 2, 3, 4, 5$. (*Hint:* Use Cauchy integral formula for unbounded domain.)

NOTE that the "finite" Cauchy's formula (more exactly, its derivative version that we will cover in the next class) is also usable, but much messier in this case.